

Static Exact Solutions of a Spin Model Exhibiting Glassy Dynamics

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A spin model which exhibits glassy dynamics has been proposed by Newman and Moore. This model possesses no randomness for exchange interactions. We propose partial trace methods for obtaining static exact solutions of this model under free boundary conditions, and show some static exact solutions. We obtain that, from the exact solutions of the specific heat and the correlation functions, the transition temperature is zero, although the thermal average of a spin is zero when the temperature is zero. We investigate the ground states. In the ground states, a part of the present results especially disagrees with the part of the previous results. We discuss the disagreements.

§1. Introduction

Spin glass models,^{2),3)} that are known as popular models for glassy systems, have randomness for exchange interactions. The mathematical study is hard. In addition, there is a problem whether the randomness for exchange interactions is necessity or not for glassy properties. Therefore, searching and investigating models, that have the properties of glassy systems and have no randomness for exchange interactions, can be one of promising methods for understandings of glassy systems. For achieving this purpose, a spin model which has no randomness for exchange interactions and exhibits glassy dynamics has been proposed by Newman and Moore.¹⁾ For dynamical features, this model has aging behavior,¹⁾ and it falls out of equilibrium at a temperature which decreases logarithmically as a function of the cooling time,¹⁾ for example. Dynamical features of this model are investigated in Refs. 1), 4), 5), 6).

For static solutions, the solutions have been exactly obtained on lattices of length a power of two along at least one dimension under periodic boundary conditions.^{1),4),5),6)} In other words, this model has a special property. Therefore, this model has been solved in a special case that, under periodic boundary conditions, the lattice length is a power of two along at least one dimension. There have been no solutions on lattices of any length under periodic boundary conditions. In addition, there have been no solutions on lattices of any length under free boundary conditions. Since the previous method for solving this model uses a special property of this model, there is a possibility that a part of the results is different from that on lattices of any length, and this issue has not been shown exactly. In this article, by solving exactly this model on lattices of any length under free boundary conditions, we exactly show that a part of the results is different from the part of the previous results.

We propose partial trace methods for obtaining static exact solutions of this model under free boundary conditions, and find some static exact solutions. We find the exact solutions under free boundary conditions for the energy, the specific heat, the free energy, the entropy, the number of ground-state degeneracy, the ther-

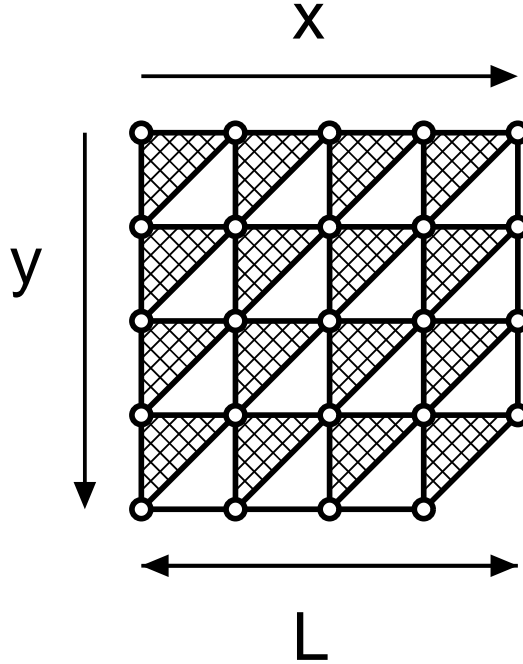


Fig. 1. The present model. In this figure, the linear size of this system, L , is 5, the number of spins, N , is 24, and the number of plackets, N_P , is 16. The plackets are drawn as meshes.

mal average of a spin, the two-point correlation function, the three-point correlation function and the four-point correlation function. We examine the transition temperature, and investigate the ground states. We discuss the disagreements between a part of the present results and the part of the previous results.

This article is organized as follows. First in §2, the model is explained. In §3, we describe the present partial trace methods for solving this model and obtain results. In §4, the disagreements between a part of the present results and the part of the previous results are discussed. This article is summarized in §5.

§2. Model

The Hamiltonian of this model under free boundary conditions, \mathcal{H} , is given by

$$\mathcal{H} = J \sum_{x=1}^{L-1} \sum_{y=1}^{L-1} \sigma_{x,y} \sigma_{x+1,y} \sigma_{x,y+1}, \quad (2.1)$$

where $\sigma_{x,y}$ is an Ising spin, $\sigma_{x,y} = \pm 1$, and x and y are coordinates of the spin. We set $J > 0$. This system is on downward pointing-triangles of the triangular lattice. L is the linear size of this system. The number of spins, N , is $L^2 - 1$, and the number of plackets, N_P , is $(L - 1)^2$.

Figure 1 shows the present model.⁷⁾ In this figure, the linear size of this system, L , is 5, the number of spins, N , is 24, and the number of plackets, N_P , is 16. The

plackets are drawn as meshes.

Note that this model is distinct from the model of Baxter and Wu,⁸⁾ which has interactions on upward-pointing triangles also. Its spin behavior of the model of Baxter and Wu is entirely different from this model.¹⁾

In the ground state, there is a relation among spins given by¹⁾

$$\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1} = -1. \quad (2.2)$$

From this relation, if the boundary spins of system are given, all the spins in the ground state are uniquely obtained.

This model displays glassy behavior under single-spin-flip dynamics. It has been pointed out that, in Ref. 1), this model has aging behavior, and it falls out of equilibrium at a temperature which decreases logarithmically as a function of the cooling time. See Refs. 1), 4), 5), 6) for dynamical features of this model.

The transformed models by a transformation $J \rightarrow \pm J$ for each three-spin interaction have identical behavior with this model.¹⁾

This model under a magnetic field has been also proposed concerned with thermodynamic transitions associated with irregularly ordered ground states.⁹⁾

§3. Results

We describe the present partial trace methods for solving this model and obtain results.

We adopt the free boundary conditions as the boundary conditions. Therefore, there is a spin which is only on a placket. We perform the integration of the spin at first. We define the spin as σ_i , and define a function $A(\sigma_j, \sigma_k, \{\sigma_l\})$, where $\{\sigma_l\}$ is the set of all spins except for σ_i , σ_j and σ_k . We introduce a variable K and define $K \equiv \beta J$, where β is the inverse temperature, and $\beta = 1/k_B T$. k_B is the Boltzmann constant, and T is the temperature. Then, we perform the integration of the spin σ_i as

$$\begin{aligned} & \sum_{\sigma_i} \sum_{\sigma_j} \sum_{\sigma_k} e^{-K\sigma_i\sigma_j\sigma_k} A(\sigma_j, \sigma_k, \{\sigma_l\}) \\ &= \sum_{\sigma_j} \sum_{\sigma_k} (e^{-K\sigma_j\sigma_k} + e^{K\sigma_j\sigma_k}) A(\sigma_j, \sigma_k, \{\sigma_l\}) \\ &= (e^{-K} + e^K) \sum_{\sigma_j} \sum_{\sigma_k} A(\sigma_j, \sigma_k, \{\sigma_l\}). \end{aligned} \quad (3.1)$$

In Eq. (3.1), we are able to treat σ_j and σ_k as the spins on the free boundaries. Therefore, we are also able to perform the integrations of σ_j and σ_k by using similar integrations with the integration of σ_i after the integration of σ_i is performed.

Figure 2 shows the partial trace method for Eq. (3.1). Figure 2(a) shows the figure before the integrations are performed. Figure 2(b) shows that the integration of the spin i is performed. Figure 2(c) shows that the integrations of the spins i and j are performed.

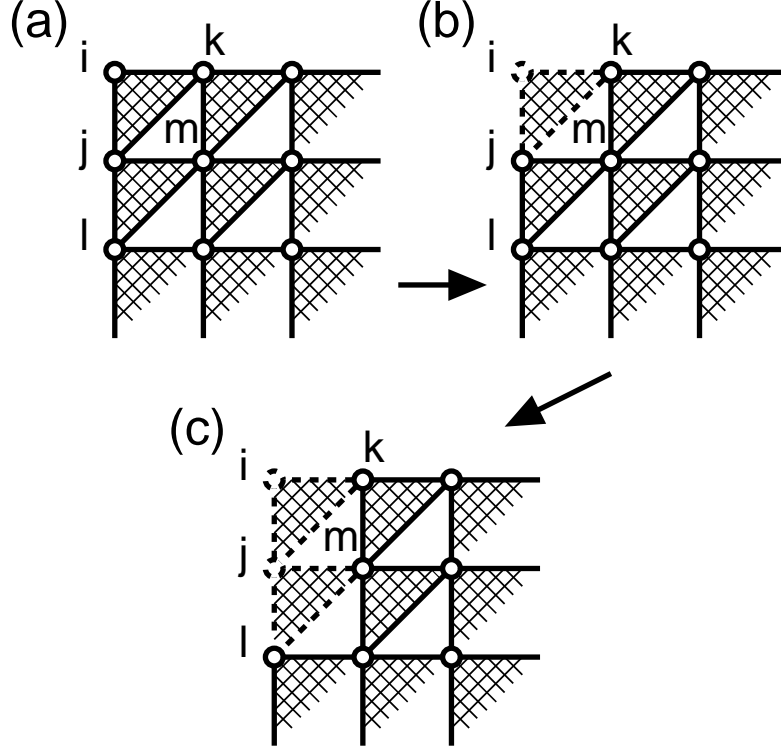


Fig. 2. The partial trace method for Eq. (3.1). (a) the figure before the integrations are performed. (b) the integration of the spin i is performed. (c) the integrations of the spins i and j are performed.

By this partial trace method for Eq. (3.1), we are able to perform the integration of the partition function Z . We obtain

$$\begin{aligned} Z &= \sum_{\{\sigma_{x,y}\}} e^{-K \sum_{x=1}^{L-1} \sum_{y=1}^{L-1} \sigma_{x,y} \sigma_{x+1,y} \sigma_{x,y+1}} \\ &= 2^{N-N_P} (e^K + e^{-K})^{N_P}. \end{aligned} \quad (3.2)$$

By using Eq. (3.2), we obtain

$$Z = 2^{2(L-1)} (e^K + e^{-K})^{(L-1)^2}. \quad (3.3)$$

By using Eq. (3.3), we obtain the energy E as

$$E = -\frac{\partial}{\partial \beta} \ln Z = -(L-1)^2 J \tanh K. \quad (3.4)$$

The energy per site in the thermodynamic limit is obtained as $-J \tanh K$. This result agrees with the previous result¹⁾ except for a constant and the coefficient of J . By using Eq. (3.4), we obtain the specific heat C as

$$C = k_B \beta^2 \frac{\partial^2}{\partial \beta^2} \ln Z = k_B K^2 (L-1)^2 \operatorname{sech}^2 K. \quad (3.5)$$

By using Eq. (3.3), we obtain the free energy F as

$$F = -\frac{1}{\beta} \ln Z = -\frac{J}{K} [2(L-1) \ln 2 + (L-1)^2 \ln(e^K + e^{-K})]. \quad (3.6)$$

By using Eqs. (3.4) and (3.6), we obtain the entropy S as

$$\begin{aligned} S &= \frac{k_B K}{J} (E - F) \\ &= 2(L-1)k_B \ln 2 + k_B(L-1)^2 [\ln(e^K + e^{-K}) - K \tanh K]. \end{aligned} \quad (3.7)$$

We define the ground-state entropy per site in the thermodynamic limit as $s^{(g)}$. We obtain

$$s^{(g)} = \lim_{K \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{S}{N} = 0. \quad (3.8)$$

The number of ground-state degeneracy, Ω , is obtained as

$$\Omega = \lim_{K \rightarrow \infty} e^{S/k_B} = 2^{2(L-1)}. \quad (3.9)$$

Therefore, the number of ground-state degeneracy is not one, although $s^{(g)} = 0$.

We calculate the thermal average of a spin, $\langle \sigma_{x,y} \rangle_T$. At first, we perform the integrations of all the spins except for the focused spin $\sigma_{x,y}$ by applying the partial trace method for Eq. (3.1). Therefore, we obtain

$$\langle \sigma_{x,y} \rangle_T = 0. \quad (3.10)$$

The present result shows $\langle \sigma_{x,y} \rangle_T = 0$ even when $T = 0$.

We calculate the two-point correlation function $\langle \sigma_{x,y} \sigma_{x',y'} \rangle_T$. At first, we perform the integrations of all the spins except for the focused spins $\sigma_{x,y}$ and $\sigma_{x',y'}$ by applying the partial trace method for Eq. (3.1). Therefore, we obtain

$$\langle \sigma_{x,y} \sigma_{x',y'} \rangle_T = \delta_{x,x'} \delta_{y,y'}, \quad (3.11)$$

where $\delta_{x,x'}$ is the Kronecker delta.

When calculations of more than three-spin correlations, we perform the integration of the spin σ_i , which is one of the focused spins for thermal average, as

$$\begin{aligned} &\sum_{\sigma_i} \sum_{\sigma_j} \sum_{\sigma_k} \sigma_i e^{-K \sigma_i \sigma_j \sigma_k} A(\sigma_j, \sigma_k, \{\sigma_l\}) \\ &= \sum_{\sigma_j} \sum_{\sigma_k} (e^{-K \sigma_j \sigma_k} - e^{K \sigma_j \sigma_k}) A(\sigma_j, \sigma_k, \{\sigma_l\}) \\ &= (e^{-K} - e^K) \sum_{\sigma_j} \sum_{\sigma_k} \sigma_j \sigma_k A(\sigma_j, \sigma_k, \{\sigma_l\}). \end{aligned} \quad (3.12)$$

If $A(\sigma_j, \sigma_k, \{\sigma_l\})$ is a Boltzmann factor for σ_j, σ_k and $\{\sigma_l\}$, we are also able to perform the integrations of σ_j and σ_k by using similar integrations with the integration of σ_i after the integration of σ_i is performed. The partial trace methods for Eqs. (3.1) and (3.12) are the partial trace methods that we propose in this article.

By applying the partial trace methods for Eqs. (3.1) and (3.12), we obtain

$$\langle \sigma_{x,y} \sigma_{x+1,y} \sigma_{x,y+1} \rangle_T = -\tanh K. \quad (3.13)$$

Similarly, we obtain

$$\begin{aligned} & \langle \sigma_{x,y} \sigma_{x+1,y} \sigma_{x,y+1} \sigma_{x',y'} \sigma_{x'+1,y'} \sigma_{x',y'+1} \rangle_T \\ &= (\tanh K)^2 + [1 - (\tanh K)^2] \delta_{x,x'} \delta_{y,y'}. \end{aligned} \quad (3.14)$$

We calculate the three-point correlation function and the four-point correlation function. The three-point correlation function has been already derived by a different method,⁴⁾ and the four-point correlation function has not been derived. We are able to consider the inside surrounded by the focused spins for thermal average and the outside of the spins separately. By using two functions A_{outside} and A_{inside} , we define

$$A_{\text{outside}} A_{\text{inside}} \equiv \langle \prod_{\{(i,j)\}} \sigma_{i,j} \rangle_T Z, \quad (3.15)$$

where $\{(i,j)\}$ is the set of coordinates of the focused spins for thermal average. A_{outside} is a weight for outside of the focused spins, and A_{inside} is a weight for inside surrounded by the focused spins. We define the number of spins for inside surrounded by the focused spins as n , and define the number of plackets for inside surrounded by the focused spins as n_P . By applying the partial trace method for Eq. (3.1), we obtain

$$A_{\text{outside}} = 2^{N-N_P-n+n_P} (e^K + e^{-K})^{N_P-n_P}. \quad (3.16)$$

We define the number of applications of the partial trace method of Eq. (3.1) for inside surrounded by the focused spins as n_1 , and define the number of applications of the partial trace method of Eq. (3.12) for inside surrounded by the focused spins as n_2 . There is a relation, written as

$$n_P = n_1 + n_2. \quad (3.17)$$

If there is a correlation for the focused spins, we obtain A_{inside} as

$$A_{\text{inside}} = 2^{n-n_1-n_2} (e^K + e^{-K})^{n_1} (e^{-K} - e^K)^{n_2}. \quad (3.18)$$

If there is no correlation for the focused spins, we obtain A_{inside} as

$$A_{\text{inside}} = 0. \quad (3.19)$$

At first, we calculate A_{outside} . The calculation of A_{outside} is started from one of the spins on the free boundaries. A_{outside} is calculated by applying the partial trace method for Eq. (3.1). Next, we calculate A_{inside} . The calculation of A_{inside} is started from one of the focused spins for thermal average. A_{inside} is calculated by applying the partial trace methods for Eqs. (3.1) and (3.12). In a certain case, the partial trace method for Eq. (3.1) is not applied to the calculation of A_{inside} , while the partial trace method for Eq. (3.12) is applied to the calculation of A_{inside} . We

find the spin correlations by examining the equations after the partial trace method for Eq. (3.12) is applied.

For the three-point correlation function, we generally obtain

$$\langle \sigma_{x,y} \sigma_{x',y'} \sigma_{x'',y''} \rangle_T = 0. \quad (3.20)$$

On the other hand, when (x, y) , (x', y') and (x'', y'') have a special relation, there is a correlation. When coordinates of three spins are (x, y) , $(x + 2^k, y)$, $(x, y + 2^k)$ and $k = 0, 1, 2, \dots$, there are correlations. Then, by applying the partial trace methods for Eqs. (3.1) and (3.12), we obtain $n = (2^{k-1} + 1)(2^k + 1)$, $n_1 = 2^{k-1}(2^k + 1) - 3^k$ and $n_2 = 3^k$. By using Eqs. (3.2), (3.15) - (3.18), we obtain

$$\langle \sigma_{x,y} \sigma_{x+2^k,y} \sigma_{x,y+2^k} \rangle_T = -(\tanh K)^{3^k}, \quad k = 0, 1, 2, \dots \quad (3.21)$$

This result agrees with the previous result obtained in Ref. 4).

For the four-point correlation function, we generally obtain

$$\langle \sigma_{x,y} \sigma_{x',y'} \sigma_{x'',y''} \sigma_{x''',y'''} \rangle_T = 0. \quad (3.22)$$

On the other hand, when (x, y) , (x', y') , (x'', y'') and (x''', y''') have a special relation, there is a correlation. When coordinates of four spins are (x, y) , $(x + 3 \cdot 2^k, y)$, $(x, y + 3 \cdot 2^k)$, $(x + 2^k, y + 2^k)$ and $k = 0, 1, 2, \dots$, there are correlations. Then, by applying the partial trace methods for Eqs. (3.1) and (3.12), we obtain $n = (3 \cdot 2^{k-1} + 1)(3 \cdot 2^k + 1)$, $n_1 = 3 \cdot 2^{k-1}(3 \cdot 2^k + 1) - 6 \cdot 3^k$ and $n_2 = 6 \cdot 3^k$. By using Eqs. (3.2), (3.15) - (3.18), we obtain

$$\langle \sigma_{x,y} \sigma_{x+3 \cdot 2^k,y} \sigma_{x,y+3 \cdot 2^k} \sigma_{x+2^k,y+2^k} \rangle_T = (\tanh K)^{6 \cdot 3^k}, \quad k = 0, 1, 2, \dots \quad (3.23)$$

For the four-spin correlations, there are positive correlations. These positive correlations for spins have not been mentioned in the previous articles.

From the solutions of the specific heat and the correlation functions, i.e., from Eqs. (3.5), (3.11), (3.20) - (3.23), the transition temperature of this model is zero, although $\langle \sigma_{x,y} \rangle_T = 0$ even when $T = 0$. This result, that the transition temperature is zero, is concluded from the thermal dependence of $\langle \sigma_{x,y} \rangle_T$ in Ref. 4). Note that, in Ref. 4), $\langle \sigma_{x,y} \rangle_T = -1$ when $T = 0$ and $\langle \sigma_{x,y} \rangle_T = 0$ when $T > 0$ are obtained.

We investigate the spin configuration on a placket. We define the probability that the spin configuration $(\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1})$ on a placket is $(-1, -1, -1)$ as $P_{x,y}(-1, -1, -1)$. $P_{x,y}(-1, -1, -1)$ is given by

$$P_{x,y}(-1, -1, -1) = \frac{1}{8}(1 - \sigma_{x,y})(1 - \sigma_{x+1,y})(1 - \sigma_{x,y+1}). \quad (3.24)$$

We obtain

$$\langle P_{x,y}(-1, -1, -1) \rangle_T = \frac{e^{2K}}{4(e^{2K} + 1)}. \quad (3.25)$$

We define the probability that the spin configuration $(\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1})$ on a placket is $(-1, 1, 1)$ as $P_{x,y}(-1, 1, 1)$. $P_{x,y}(-1, 1, 1)$ is given by

$$P_{x,y}(-1, 1, 1) = \frac{1}{8}(1 - \sigma_{x,y})(1 + \sigma_{x+1,y})(1 + \sigma_{x,y+1}). \quad (3.26)$$

We obtain

$$\langle P_{x,y}(-1, 1, 1) \rangle_T = \frac{e^{2K}}{4(e^{2K} + 1)}. \quad (3.27)$$

Similarly, we obtain

$$\langle P_{x,y}(1, -1, 1) \rangle_T = \langle P_{x,y}(1, 1, -1) \rangle_T = \frac{e^{2K}}{4(e^{2K} + 1)}. \quad (3.28)$$

When $K \rightarrow \infty$, $\langle P_{x,y} \rangle_T$ becomes the probability in the ground states. Therefore, in the ground states, the spin configurations $(-1, -1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ emerge with each 1-in-4 chance.

§4. Discussion

There are disagreements between a part of the present results and the part of the previous results. We discuss the disagreements here.

The present result shows that the number of ground-state degeneracy is $2^{2(L-1)}$. On the other hand, in Ref. 1), it has been pointed out that the number of ground-state degeneracy is one.

The present result shows $\langle \sigma \rangle_T = 0$ even when $T = 0$. In addition, the probability that the spin configuration $(\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1})$ on a plaquette is $(-1, -1, -1)$ is $1/4$ when $T = 0$. On the other hand, in Refs. 1), 4), it has been pointed out that $\langle \sigma \rangle_T = -1$ when $T = 0$.

This model has a special property. This model has been exactly solved in a special case that, under periodic boundary conditions, the lattice length is a power of two along at least one dimension, and has not been exactly solved on lattices of any length, so that there is a possibility that a part of the results on lattices of any length is different from the part of the previous results. From the results on lattices of any length under free boundary conditions, we see that there are the differences. Especially, the results for the ground states are different from the previous results as described above.

These disagreements between the present results and the previous results would not seriously affect the dynamics. The dynamics are decided by the energy changes of spin-flips,¹⁾ and the energy changes of spin-flips are not mentioned in this article, but the energy changes of spin-flips are the same with the previous results. The number of ground-state degeneracy is different from that of the previous result, but the ground state entropy in the thermodynamic limit is the same with the previous result, so that the disagreement for ground-state degeneracy would not seriously affect the dynamics. Since the dynamics are decided by the energy changes of spin-flips, the effect from the different boundary conditions would not seriously affect the dynamics if the system size is large enough.

§5. Summary

A spin model, which exhibits glassy dynamics and has no randomness for exchange interactions, has been proposed by Newman and Moore. We proposed partial

trace methods for obtaining static exact solutions of this model under free boundary conditions, and showed some static exact solutions. The partial trace methods for Eqs. (3·1) and (3·12) are the partial trace methods that we proposed in this article. We obtained that, from the exact solutions of the specific heat and the correlation functions, the transition temperature is zero, although the thermal average of a spin is zero when the temperature is zero. We investigated the ground states. In the ground states, a part of the present results especially disagreed with the part of the previous results. We discussed the disagreements, and concluded that the disagreements would not seriously affect the dynamics.

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